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Abstract

A labeling of the vertices of a graph by elements of any abelian group A induces a labeling of the edges by summing the labels of their endpoints. Hovey defined the graph G to be A-cordial if it has such a labeling where the vertex labels and the edge labels are both evenly-distributed over A in a technical sense. His conjecture that all trees T are A-cordial for all cyclic groups A remains wide open, despite significant attention. Curiously, there has been very little study of whether Hovey's conjecture might extend beyond the class of cyclic groups.

We initiate this study by analyzing the larger class of finite abelian groups A such that all path graphs are A-cordial. We conjecture a complete characterization of such groups, and establish this conjecture for various infinite families of groups as well as for all groups of small order.

1 Introduction

Let A be a finite abelian group of order n. A labeling of the vertices of any graph G by elements of A induces a labeling of the edges of G by associating to each edge the sum of the labels on its endpoints. Hence, every such vertex labeling gives rise to two integer partitions, each with at most n parts: $\lambda^v := (\lambda_1^v \geqslant \cdots \geqslant \lambda_n^v \geqslant 0)$ recording the number of vertices with each label and $\lambda^e := (\lambda_1^e \geqslant \cdots \geqslant \lambda_n^e \geqslant 0)$

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recording the number of edges with each label (both arranged in decreasing order). We say a partition $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \cdots)$ is ***almost rectangular*** if, for all i, we have $\lambda_i \in \{\lambda_1, \lambda_1 - 1\}$.

In early work, Hovey [6] introduced the following notion.

Definition 1.1. A graph G is A***-cordial*** if there is a vertex labeling of G such that both partitions λ^v, λ^e are almost rectangular.

Hovey's definition subsumes other famous notions in graph labeling. For example, the graph G is $\mathbb{Z}_2$-cordial if and only if it is *cordial* in the sense of Cahit [1], while G is $\mathbb{Z}_{|E(G)|}$-cordial if and only if it is *harmonious* in the sense of Graham and Sloane [5]. (Here $\mathbb{Z}_k$ denotes the cyclic group of order k, written additively.) For an extensive survey of graph labeling, see [4].

Most work to date has taken the form of fixing a group A (nearly always cyclic) and asking which graphs G are A-cordial. Results along these lines, while numerous, are mostly piecemeal and having *ad hoc* proofs. Here, we consider a dual problem.

Definition 1.2. Let $\mathbb{G}$ be a family of graphs. We say a group A is $\mathbb{G}$***-cordial*** if every $G \in \mathbb{G}$ is A-cordial. We say A is ***weakly*** $\mathbb{G}$***-cordial*** if all but finitely many $G \in \mathbb{G}$ are A-cordial.

A major open problem is Hovey's conjecture [6, Conjecture 1] that all trees are A-cordial for all cyclic groups A. In other language, we have the following.

Conjecture 1.3 ([6])**.** Let $\mathbb{T}$ be the class of trees. Then $\mathbb{Z}_k$ is $\mathbb{T}$-cordial for all k.

Currently, the only nontrivial finite groups known to be $\mathbb{T}$-cordial are the small cyclic groups $\mathbb{Z}_2$ [1], $\mathbb{Z}_3$ [6], $\mathbb{Z}_4$ [6], $\mathbb{Z}_5$ [6], $\mathbb{Z}_6$ [3], and $\mathbb{Z}_7$ [2]. We wish to gain insight into Conjecture 1.3 by considering the class of $\mathbb{T}$-cordial groups more broadly. Certainly, not all abelian groups are $\mathbb{T}$-cordial, as it is easy to check that the four-vertex path P_4 is not $\mathbb{Z}_2 \times \mathbb{Z}_2$-cordial. (All group products in this paper are direct products.) However, it is currently unknown whether any finite noncyclic group is $\mathbb{T}$-cordial; indeed, there are not even any conjectures in this direction. The results of this paper, however, may be taken as evidence that many finite noncyclic groups are also $\mathbb{T}$-cordial. Hence, the appropriate level of generality for studying Conjecture 1.3 may be broader than the class of cyclic groups.

It seems reasonable to start this program by considering the broader family of $\mathbb{P}$-cordial groups, where $\mathbb{P}$ denotes the class of *path graphs* (i.e., trees with no vertex of degree > 2). Hovey [6, Theorem 2] showed that all cyclic groups are $\mathbb{P}$-cordial, whereas $\mathbb{Z}_2 \times \mathbb{Z}_2$ is not. To our knowledge, no other results about $\mathbb{P}$-cordiality have been established to date. We make the following conjecture.

Conjecture 1.4. A finite abelian group A is $\mathbb{P}$-cordial if and only if it is not a nontrivial product of copies of $\mathbb{Z}_2$ (equivalently, if and only if there exists $a \in A$ with $|a| > 2$).

Our main results are evidence toward Conjecture 1.4. The following proves one direction.

Theorem 1.5. *If $A = \mathbb{Z}_2^m$ is a product of copies of $\mathbb{Z}_2$ ($m > 1$), then P_{2m} and P_{2m+1} are not A-cordial (and so A is not $\mathbb{P}$-cordial).*

For the other direction of Conjecture 1.4, our main result is the following.

Theorem 1.6. *If $|A|$ is odd, then A is $\mathbb{P}$-cordial.*

It remains to understand the case of groups A that are of even order and have an element of order greater than 2. We give some partial results in that setting. We also verify Conjecture 1.4 for all abelian groups A with $|A| < 24$.

While $\mathbb{Z}_2^2$ is not $\mathbb{P}$-cordial, it is known to be *weakly* $\mathbb{P}$-cordial [7, Theorem 3.4]. We show that $\mathbb{Z}_2^3$ is similarly weakly $\mathbb{P}$-cordial, while not $\mathbb{P}$-cordial. On the basis of these examples, [7, Corollary 4.3] (showing that there are infinitely-many A-cordial paths for each A), and Conjecture 1.4, we also expect the following to hold.

Conjecture 1.7. All finite abelian groups are weakly $\mathbb{P}$-cordial.

This paper is structured as follows. In Section 2, we prove some results that hold for all finite abelian groups A and that we will need in later sections. In particular, Theorem 2.4 allows us to demonstrate $\mathbb{P}$-cordiality of A by showing the A-cordiality of a single path. In Section 3, we apply Theorem 2.4 to prove Theorem 1.6, showing that all A of odd order are $\mathbb{P}$-cordial. In Section 4, we study products of groups of order two, proving Theorem 1.5 and verifying that $\mathbb{Z}_2^3$ is weakly $\mathbb{P}$-cordial. Finally, Section 5 is devoted to other groups of even order, including a complete analysis of $\mathbb{P}$-cordiality for all abelian groups of small order.

2 Results for general groups

Here we collect various results that hold for all finite abelian groups A. We use these results later to establish our main results.

Lemma 2.1. *Let A be any abelian group of order n. Let $f, k \in \mathbb{Z}_{\geqslant 0}$ be nonnegative integers such that $f \leqslant \frac{n}{2}$. Then if the path P_{nk+f} is A-cordial, so is P_{nk+f+1}.*

Proof. This is a generalization of [6, Lemma 3] and the proof is essentially the same, but we include the details for completeness.

Consider an A-cordial labeling of P_{nk+f} with almost rectangular partitions λ^v, λ^e. Attach a new vertex x to one of the ends of the path by a new edge y. It suffices to exhibit an appropriate vertex label for x. There are $n - f$ available vertex labels for it that would keep λ^v almost rectangular.

If $f = 0$, then there is a unique label for y that makes λ^e almost rectangular, and we may choose the label for x accordingly.

If $f > 0$, there are only $f-1$ edge labels on y that would make λ^e no longer almost rectangular. So we can find an appropriate label for x provided that $f-1 < n-f$, that is if $2f < n+1$. □

Lemma 2.2. *Suppose G is an A-cordial graph and $a \in A$. Then for any A-cordial labeling of G, we may add a to each vertex label to obtain another A-cordial labeling.*

Proof. This is [6, Lemma 1]. □

Lemma 2.3. *Suppose $|A| = n$ and let $k, m \in \mathbb{Z}_{>0}$ be positive integers. If P_k and P_{mn} are both A-cordial, then so is P_{mn+k}.*

Proof. Consider A-cordial labelings of P_k and P_{mn}. Every element of A appears exactly m times as a vertex label of P_{mn}. There is a unique element $a \in A$ that appears $m-1$ times as an edge label of P_{mn}, while all others appear exactly m times. By Lemma 2.2, we may assume that P_{mn} has an endpoint labeled by the identity element id and that P_k has an endpoint labeled a. Join the id-end of P_{mn} to the a-end of P_k by a new edge (necessarily labeled a) to obtain a labeled P_{mn+k}.

Then we have added to P_k exactly m of each vertex label and exactly m of each edge label. That is, for all $1 \leqslant i \leqslant n$, we have $\lambda_i^v(P_{mn+k}) = \lambda_i^v(P_k) + m$ and $\lambda_i^e(P_{mn+k}) = \lambda_i^e(P_k) + m$. Since the labeling of P_k was A-cordial, it follows that this labeling of P_{mn+k} is as well. □

Theorem 2.4. *Suppose $|A| = n$. Then A is $\mathbb{P}$-cordial if and only if P_n is A-cordial.*

Proof. One implication is trivial. For the other, suppose P_n is A-cordial. Then iteratively deleting vertices from either end of an A-cordial labeling of P_n, we see that P_k is A-cordial for all $k < n$.

Now consider any path P_m. Write $m = hn + k$ for some $h, k \in \mathbb{Z}$ and $0 \leqslant k < n$. Then we may construct an A-cordial labeling of P_m by gluing h A-cordial labelings of P_n to an A-cordial labeling of P_k, as in Lemma 2.3. □

The previous theorem makes it easy to demonstrate that a group A is $\mathbb{P}$-cordial because it is thereby sufficient to exhibit a single A-cordial labeling of a single graph. For any fixed A, Theorem 2.4 indeed makes it a finite check to determine whether A is $\mathbb{P}$-cordial. In practice, however, this exhaustive check is not feasible for groups A that are not of extremely small order.

3 Groups of odd order

In this section, we prove Theorem 1.6, showing that all groups of odd order are $\mathbb{P}$-cordial. The reader may find the technical details of the proof clarified by consulting the example for $\mathbb{Z}_3 \times \mathbb{Z}_3$ displayed in Figure 1.

Lemma 3.1. *Suppose $|A| = n$ and the cycle C_n is A-cordial. Then, for k odd, the cycle C_{kn} is $A \times \mathbb{Z}_k$-cordial.*

Proof. Consider an A-cordial labeling of C_n and write $\ell(v)$ for the label of vertex v. Fix a cyclic orientation of C_n. Subdivide each side of C_n by inserting $k-1$ new vertices into each edge of C_n to obtain a cycle C_{kn}. Each (directed) edge $q = \overrightarrow{xy}$ of C_n with endpoints x, y becomes an induced (oriented) path P_{k+1} with vertices $x \to q_1 \to \cdots \to q_{k-1} \to y$. Label these vertices alternately $\ell(x)$ and $\ell(y)$, so that x is still labeled $\ell(x)$ and y is still labeled $\ell(y)$. More precisely, we define $\ell(q_i) = \ell(x)$ if i is even and $\ell(q_i) = \ell(y)$ if i is odd.

We claim that this process results in an A-cordial labeling of C_{kn}. Note that if x has neighbors y, z in C_n, then in our C_{kn} the label $\ell(x)$ appears on $\frac{k-1}{2} + 1$ vertices along the path from x to y and on $\frac{k-1}{2} + 1$ vertices along the path from z to x, with the vertex x being shared. Hence, exactly k vertices of C_{kn} are labeled $\ell(x)$. Since we started with a labeling of C_n with every element of A appearing exactly once as a vertex label, this means that the vertex labels of our C_{kn} yield an almost rectangular partition.

All of the edges along the path from x to y in C_{kn} are labeled $\ell(x) + \ell(y)$. Hence the edge labels of our C_{kn} are the edge labels of the original C_n in the same order, but each now repeated k times in a row. Since each element of A appeared exactly once as an edge label of C_n, this means that the edge labels of the C_{kn} also give an almost rectangular partition, so our construction is an A-cordial labeling of C_{kn}.

Finally, we will convert this labeling of C_{kn} into an $A \times \mathbb{Z}_k$-cordial labeling. For each of the vertices x from C_n, label x by the ordered pair $(\ell(x), 0) \in A \times \mathbb{Z}_k$. Along the oriented path $x \to q_1 \to \cdots \to q_{k-1} \to y$, label each vertex q_{2i} by $(\ell(x), i + \frac{k-1}{2}) \in A \times \mathbb{Z}_k$ and each vertex q_{2i+1} by $(\ell(y), i) \in A \times \mathbb{Z}_k$.

It is straightforward to see that each element of $A \times \mathbb{Z}_k$ appears now as a vertex label exactly once. Moreover, the second coordinates of the edge labels rotate cyclically through the elements of $\mathbb{Z}_k$ in cyclic order. Since each element of A appears as a first coordinate on exactly k consecutive edges, this implies that each element of $A \times \mathbb{Z}_k$ also appears as an edge label exactly once, so we have constructed an $A \times \mathbb{Z}_k$-cordial labeling of C_{nk}. $\square$

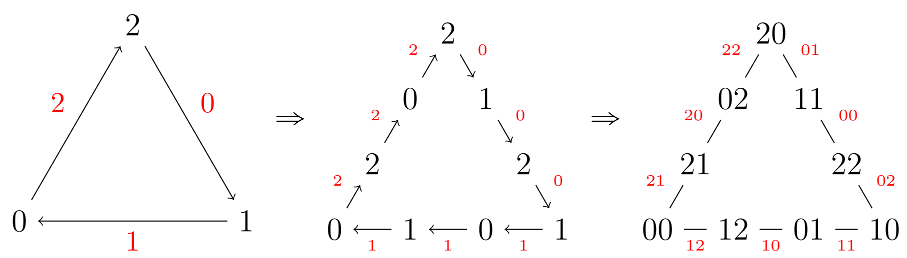

Figure 1: Example of constructing a $\mathbb{Z}_3 \times \mathbb{Z}_3$-cordial labeling of C_9 from a $\mathbb{Z}_3$-cordial labeling of C_3, as described in the proof of Lemma 3.1.

Lemma 3.2. *Suppose $|A| = n$ is odd. Then C_n is A-cordial.*

Proof. By induction on n. If A is a cyclic group, this is [6, Theorem 9]. Otherwise, write A as $B \times \mathbb{Z}_k$ for some odd k. Then $|B| = b$ is also odd, so by induction, C_b is B-cordial. Therefore Lemma 3.1 gives that C_{kb} is $B \times \mathbb{Z}_k$-cordial, i.e., C_n is A-cordial. □

Proof of Theorem 1.6. Let $|A| = n$ be odd. Then by Lemma 3.2, the cycle C_n is A-cordial. Take an A-cordial labeling of C_n. Deleting any edge yields an A-cordial labeling of the path P_n. Since P_n is A-cordial, Theorem 2.4 then implies that A is $\mathbb{P}$-cordial, as desired. □

4 Products of order-2 groups

In this section, we assume $A = \mathbb{Z}_2^m$ for some $m > 1$. We first prove Theorem 1.5, showing that such an A cannot be $\mathbb{P}$-cordial.

Proof of Theorem 1.5. Suppose we had an A-cordial labeling of P_{2^m}. Then each $a \in A$ appears exactly once as a vertex label, and exactly one element of A does not appear as an edge label. This missing element must be id, since an edge labelled id can only come from adjacent vertices labeled a and $-a$, but every element of A is its own inverse. Therefore, the edge labels are the $2^m - 1$ nonidentity elements of A, each appearing exactly once.

It is easy to see that the sum of all the (nonidentity) elements of A is id. Hence, the sum of the edge labels is id. On the other hand, the edge labels come from adding adjacent vertex labels. Thus, the sum of the edge labels is the sum of the two leaf vertex labels plus twice the labels of all internal vertices. However, twice any group element is the identity, so the sum of the edge labels is just the sum of the two leaf vertex labels. Hence, the labels on the leaf vertices are each other's inverses, contradicting that they must be distinct. Therefore, P_{2^m} is not A-cordial.

Now, suppose we had an A-cordial labeling of P_{2^m+1}. Then each $a \in A$ appears exactly once as an edge label. Let e be the edge labeled id, and let its endpoints be x, y. Then the labels on x and y sum to id and hence are equal. Contract the edge e to get a labeled P_{2^m}. It has each $a \in A$ appearing exactly once as a vertex label and each nonidentity element appearing exactly once as an edge label, contradicting that P_{2^m} is not A-cordial. □

The special case $m = 2$ was studied in [7]. By Theorem 1.5, P_4 and P_5 are not $\mathbb{Z}_2 \times \mathbb{Z}_2$-cordial. However, [7, Theorem 3.4] shows that all other paths are $\mathbb{Z}_2 \times \mathbb{Z}_2$-cordial, so that $\mathbb{Z}_2 \times \mathbb{Z}_2$ is *weakly* $\mathbb{P}$-cordial. As evidence towards Conjecture 1.7, we here establish analogous results in the case $m = 3$.

For notational convenience, let $M = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. We further identify the elements of M with binary strings of length 3, added componentwise.

Proposition 4.1. *The group M is weakly $\mathbb{P}$-cordial. More precisely, all paths are M-cordial except P_8 and P_9.*

Proof. Theorem 1.5 gives that P_8 and P_9 are not M-cordial.

Lemma 2.1 shows that P_k is M-cordial for $k \leqslant 5$. The reader may confirm that

$$100 - 000 - 001 - 010 - 111 - 101$$

is an M-cordial labeling of P_6 (here we record only the vertex labels), while

$$100 - 000 - 001 - 010 - 111 - 101 - 011$$

is an M-cordial labeling of P_7.

Additionally,

$$000 - 111 - 001 - 010 - 110 - 011 - 011 - 100 - 101 - 111$$

is an M-cordial labeling of P_{10}. Therefore, the M-cordiality of P_{11}, P_{12}, and P_{13} follows immediately by Lemma 2.1.

Further observe that the following are M-cordial labelings of P_{14}, P_{15}, and P_{16}:

$$P_{14}: \quad 000 - 111 - 001 - 010 - 110 - 011 - 011 - 100 - 101 - 111 - 001 - 010 - 110 - 100$$
$$P_{15}: \quad 000 - 111 - 001 - 010 - 110 - 011 - 011 - 100 - 101 - 111 - 001 - 010 - 110 - 100 - 101$$
$$P_{16}: \quad 000 - 000 - 111 - 001 - 010 - 110 - 011 - 011 - 100 - 101 - 111 - 001 - 010 - 110 - 100 - 101$$

Since $16 = 2 \cdot |M|$, if $k = 16m + j$ for some $0 \leqslant j < 16$ with $j \notin \{8, 9\}$, then we may construct an M-cordial labeling of P_k by Lemma 2.3, gluing an M-cordial labeling of P_j (as constructed above) to m copies of the M-cordial labeling of P_{16} given above. Hence, we have shown that P_k is M-cordial for all k not congruent to 8 or 9 modulo 16.

A M-cordial labeling of P_{24} is

000-001-101-000-111-001-010-110-011-011-100-101-111-001-010-110-100-100-011-010-111-101-110-000.

By Lemma 2.1, it then follows that P_{25} is also M-cordial. Using Lemma 2.3 to attach an appropriate number of labeled copies of P_{16}, we obtain M-cordial labelings of all remaining paths. $\square$

5 Other groups of even order

While we know from Section 3 that all groups of odd order are $\mathbb{P}$-cordial and from Section 4 that nontrivial products of order-2 groups are not, the situation for other groups of even order is more mysterious. Although Conjecture 1.4 predicts that all such groups should also be $\mathbb{P}$-cordial, we only establish limited progress in this direction, as well as verifying Conjecture 1.4 for groups of small order.

Proposition 5.1. *If A is an abelian group with $|A| < 24$ that is not a nontrivial direct product of groups of order 2, then A is $\mathbb{P}$-cordial.*

Proof. Let $n = |A|$ and assume $n < 24$. If A is cyclic, then we are done by [6, Theorem 2]. If n is odd, we are done by Theorem 1.6.

Up to group isomorphism, it remains to consider the following groups A:

(a) $\mathbb{Z}_2 \times \mathbb{Z}_4$,

(b) $\mathbb{Z}_2 \times \mathbb{Z}_6$,

(c) $\mathbb{Z}_2 \times \mathbb{Z}_8$,

(d) $\mathbb{Z}_4 \times \mathbb{Z}_4$,

(e) $\mathbb{Z}_3 \times \mathbb{Z}_6$,

(f) $\mathbb{Z}_2 \times \mathbb{Z}_{10}$.

By Theorem 2.4, it then suffices to exhibit an A-cordial labeling of P_n for each of these six groups. For compactness, we write ij to denote the group element $(i, j) \in \mathbb{Z}_a \times \mathbb{Z}_b$, and record only the vertex labels. The reader may verify directly that the following are A-cordial labelings for the corresponding groups:

(a) 00 – 12 – 10 – 01 – 02 – 03 – 11 – 13,

(b) 03 – 00 – 15 – 13 – 11 – 05 – 02 – 12 – 14 – 04 – 01 – 10,

(c) 00 – 14 – 16 – 07 – 04 – 15 – 12 – 13 – 11 – 02 – 10 – 06 – 03 – 05 – 01 – 17,

(d) 00 – 20 – 23 – 03 – 12 – 11 – 33 – 01 – 13 – 32 – 21 – 10 – 22 – 30 – 31 – 02,

(e) 00 – 25 – 21 – 01 – 02 – 22 – 12 – 15 – 04 – 11 – 24 – 20 – 10 – 13 – 05 – 03 – 23 – 14,

(f) 00 – 11 – 07 – 05 – 12 – 15 – 14 – 06 – 18 – 16 – 09 – 01 – 04 – 19 – 17 – 02 – 10 – 13 – 03 – 08.

□

The proof of Proposition 5.1 relies on ad hoc construction of some examples of A-cordial labelings. For even-order groups A that are generated by two elements, the following proposition gives a uniform construction of A-cordial paths. It seems like this construction (in combination with the lemmas of Section 2) would be useful in a proof that such groups are at least *weakly* $\mathbb{P}$-cordial, a step towards establishing Conjectures 1.4 and 1.7.

Proposition 5.2. *If* $A = \mathbb{Z}_2 \times \mathbb{Z}_k$ *and* $n = |A|$, *then* P_{2n} *is* A*-cordial.*

Proof. If k is odd, then A is a cyclic group, so all paths are A-cordial by [6, Theorem 2].

Hence assume $k = 2m$ is even. We will exhibit an explicit A-cordial labeling of P_{2n} by first building an auxiliary labeling of P_{2n} that is not A-cordial and then slightly modifying this auxiliary labeling. To start, draw P_{2n} in two rows as a bipartite graph. The second coordinates of the top-row vertices will be $0, 1, \ldots, k-1, 0, 1, \ldots, k-1$ from left to right. The second coordinates of the bottom-row vertices will also be $0, 1, \ldots, k-1, 0, 1, \ldots, k-1$ from left to right. The first coordinates of the top-row alternate between 0 and 1, starting with 1, while those of the bottom row alternate starting with 0.

It is clear that this labeling uses each element of A as a vertex label exactly twice. However, the labeling constructed so far is not A-cordial. The second coordinates of the edge labels cycle through $0, 1, \ldots, k-1, 0, 1, \ldots, k-1$ in order four times, while the first coordinates of the edge labels alternate between 0 and 1. Since k is even, this means that each $(i, j) \in \mathbb{Z}_2 \times \mathbb{Z}_k$ with $i + j$ odd appears as a label on four edges, while each $(i, j) \in \mathbb{Z}_2 \times \mathbb{Z}_k$ with $i + j$ even does not appear as an edge label.

The final step is to modify this labeling by swapping (the first coordinates of) the last m labels of the top row with (the first coordinates of) the second batch of m labels from the bottom row. This swapping is illustrated in Figure 2 for the example $A = \mathbb{Z}_2 \times \mathbb{Z}_4$.

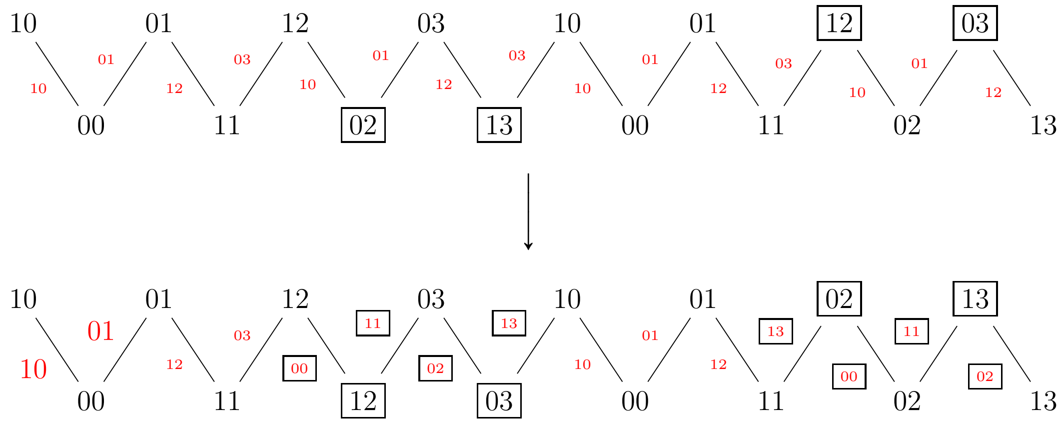

Figure 2: An example of the labeling process from the proof of Proposition 5.2, as illustrated for $A = \mathbb{Z}_2 \times \mathbb{Z}_4$. (Note that this example is only for illustrative purposes, as we already know from Theorem 5.1 that this A is $\mathbb{P}$-cordial.) The upper graph shows the auxiliary labeling with boxes around the vertex labels to be swapped. The lower graph shows the A-cordial labeling obtained by swapping, with boxes around the (vertex and edge) labels that have changed.

It is clear that this modified labeling also uses each element of A as a vertex label exactly twice. The second coordinates of the edge labels still cycle through $0, 1, \ldots, k-1, 0, 1, \ldots, k-1$ in order four times. However, the pattern of first coordinates of edge labels is now more complicated.

Edge labels appear in the following sequence from left to right (we include line breaks that we believe help clarify the structure of the sequence):

$$\begin{aligned}&10, 01, 12, 03, \ldots, 1(k-2), 0(k-1),\\ &00, 11, 02, 13, \ldots, 0(k-2), 1(k-1),\\ &10, 01, 12, 03, \ldots, 1(k-2),\\ &1(k-1),\\ &00, 11, 02, 13, \ldots, 0(k-2).\end{aligned}$$

In particular, the first k labels are exactly the same as the second batch of k labels and in the same order, except that all the first coordinates have been switched. Hence, the first n edge labels consist of each element of A exactly once. Similarly, after the first n edge labels, the next $k-1$ labels are the same as the last $k-1$ edge labels and in the same order, except that all the first coordinates have been switched. Hence we see that, in total, each label appears exactly twice as an edge label, except for the label $0(k-1)$, which appears exactly once. Thus, this is an A-cordial labeling of P_{2n}. □

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